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ON ANTI-PLANE SHEAR BEHAVIOR OF A GRIFFITH PERMEABLE CRACK IN PIEZOELECTRIC MATERIALS BY USE OF THE NON-LOCAL THEORY*

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ABSTRACT: In this paper, the non-local theory of elasticity is applied to obtain the behavior of a Griffith crack in the piezoelectric materials under anti-plane shear loading for permeable crack surface conditions. By means of the Fourier transform, the problem can be solved with the help of a pair of dual integral equations with the unknown variable being the jump of the displacement across the crack surfaces. These equations are solved by the Schmidt method. Numerical examples are provided. Unlike the classical elasticity solutions, it is found that no stress and electric displacement singularity is present at the crack tip. The non-local elastic solutions yield a finite hoop stress at the crack tip, thus allowing for a fracture criterion based on the maximum stress hypothesis. The finite hoop stress at the crack tip depends on the crack length and the lattice parameter of the materials, respectively.

KEY WORDS: crack, non-local'theory, piezoelectric materials, Fourier integral transform, Schmidt method

1 INTRODUCTION

In the theoretical studies of crack problems of the piezoelectric materials, several different electric boundary conditions at the crack surfaces have been proposed by numerous researchers[1~7]. The basic theory and the problems of dislocation, crack and inclusion of the piezoelectric materials were investigated in Ref.[1]. The electric saturation crack model in the piezoelectric materials was studied in Refs.[2,3]. A complete exact solution was obtained in Ref.[3] for a single electric saturation crack in an infinite piezoelectric media. In fact, cracks in piezoelectric materials consist of vacuum, air or some other gas. This requires that the electric fields can propagate through the crack, so the electric displacement component perpendicular to the crack surfaces should be continuous across the crack surfaces. Along this line, the crack problem of piezoelectric materials was studied in Ref.[4]. Recently, Dunn[5] and Sosa and Khutoryansky[6] avoided the common assumption of electric impermeability and utilized more accurate electric boundary conditions at the rim of an elliptical flaw to deal with anti-plane problems of piezoelectricity. They analyzed the effects of electric boundary conditions at the crack surfaces on the fracture behaviors of piezoelectric materials. Most recently, the behavior of a bi-piezoelectric ceramic layer with an interfacial crack has been investigated by using the dislocation density function and the singular integral equation method for two different crack surface boundary conditions in Ref.[7], respectively, i.e. permeable and impermeable. It is interesting to note that very different results were obtained with different boundary conditions. However, these solutions contain stress singularity. This is not reasonable according to the physical nature. The stresses near the tip of a sharp line crack in an elastic plane subject to uniform tension, shear and anti-plane shear were discussed in Refs.[8~10] by use of the non-local theory. These solutions gave finite stresses at the crack tips, thus resolving a fundamental problem that persisted for many years. This

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enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way. The solutions in Refs.[8~10], however, were not exact and were reexamined in Refs.[11,12] using a different approach. The dynamic stresses near the tip of a line crack or two line cracks in an elastic plane were investigated in Refs.[13,14] by use of non-local theory. These solutions did not contain any stress singularity. To our knowledge, the electro-elastic behavior of the piezoelectric materials with a permeable crack subjected to anti-plane shear and in-plane electric loading has not been studied by the non-local theory.

In the present paper, the behavior of a permeable crack subjected to anti-plane shear in piezoelectric materials is investigated by use of the non-local theory. The traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects. Fourier transform is applied and a mixed boundary value problem is reduced to a pair of dual integral equations with the unknown variable being the jump of the displacement across the crack surfaces. In solving the dual integral equations, the jump of the displacement across the crack surface expanded in a series of Jacobi polynomials and the Schmidt method[15] is used. This process is quite different from that adopted in previous studies[1~10] as mentioned above. As expected, the solution in this paper does not contain the stress and electric displacement singularity at the crack tip. The stress field and the electric field for the non-local theory are similar to those of the classical elasticity solution away from the crack tips. Near the crack tip, a lattice parameter tends to control the amplitude of the stress and the electric displacement.

2 BASIC EQUATIONS OF NON-LOCAL PIEZOELECTRIC MATERIALS

For the anti-plane shear problem, the basic equations of linear, homogeneous, isotropic, non-local piezoelectric materials, with vanishing body force are[4,10]

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \tag{2}$$

$$\tau_{kz}(X) = \int_V [c'_{44}(|X'-X|)w_{,k}(X') + e'_{15}(|X'-X|)\phi_{,k}(X')]\mathrm{d}V(X') \qquad k = x, y \tag{3}$$

$$D_k(X) = \int_V [e'_{15}(|X'-X|)w_{,k}(X') - \varepsilon'_{11}(|X'-X|)\phi_{,k}(X')]\mathrm{d}V(X') \qquad k = x, y \tag{4}$$

where the only difference between the classical elastic theory and the piezoelectric theory is in the stress and the electric displacement constitutive Eqs.(3), (4) in which the stress $\tau_{zk}(X)$ and the electric displacement $D_k(X)$ at a point X depends on $w_{,k}(X)$ and $\phi_{,k}(X)$, at all points of the body. w and ϕ are the mechanical displacement and electric potential. For homogeneous and isotropic piezoelectric materials there exist only three material parameters, $c'_{44}(|X'-X|)$, $e'_{15}(|X'-X|)$ and $\varepsilon'_{11}(|X'-X|)$ which are functions of the distance $|X'-X|$. The integrals in Eqs.(3), (4) are over the volume V of the body enclosed within a surface ∂V.

As discussed in Refs.[16,17], the form of $c'_{44}(|X'-X|)$, $e'_{15}(|X'-X|)$ and $\varepsilon'_{11}(|X'-X|)$ can be assumed such that the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves the following has been found to be very useful

$$(c'_{44},\, e'_{15},\, \varepsilon'_{11}) = (c_{44},\, e_{15},\, \varepsilon_{11})\alpha(|X'-X|) \tag{5}$$

$$\alpha(|X'-X|) = \alpha_0 \exp[-(\beta/a)^2(X'-X)(X'-X)] \tag{6}$$

where β is a constant, a is the lattice parameter of the materials. c_{44}, e_{15}, ε_{11} are the shear modulus, piezoelectric coefficient and dielectric parameter, respectively. α_0 is determined by the normalization

$$\int_V \alpha(|X'-X|)\mathrm{d}V(X') = 1 \tag{7}$$

In the present work, the non-local elastic moduli are given by Eqs.(5) and (6). Substituting Eq.(6) into Eq.(7), it can be obtained, in the two-dimensional space, that

$$\alpha_0 = \frac{1}{\pi}(\beta/a)^2 \tag{8}$$

Substitution of Eqs.(5), (6) into Eqs.(3), (4) yields

$$\tau_{kz}(X) = \int_V \alpha(|X'-X|)\sigma_{kz}(X')\mathrm{d}V(X') \qquad k = x, y \tag{9}$$

$$D_k(X) = \int_V \alpha(|X'-X|)D_k^c(X')\mathrm{d}V(X') \qquad k = x, y \tag{10}$$

where

$$\sigma_{kz} = c_{44} w_{,k} + e_{15}\phi_{,k} \qquad k = x, y \tag{11}$$

$$D_k^c = e_{15} w_{,k} - \varepsilon_{11}\phi_{,k} \qquad k = x, y \tag{12}$$

Expressions (11), (12) are the classical constitutive equations.

3 THE CRACK MODEL

Consider an infinite piezoelectric body containing a Griffith permeable crack of length $2l$ along the x-axis. The piezoelectric boundary-value problem for anti-plane shear is considerably simplified if we consider only the out-of-plane displacement and the in-plane electric fields as shown in Fig.1. As discussed in Refs.[7,10], since no opening displacement exists for the present anti-plane problem, the crack surfaces can be assumed to be in perfect contact. Accordingly, a permeable condition will be enforced in the present study, i.e., both the electric potential and the normal electric displacement are assumed to be continuous across the crack surfaces. So the boundary conditions of the present problem are (In this paper, we consider the perturbation stress field and the perturbation electric displacement field)

$$\tau_{yz}^{(1)}(x, 0^+) = \tau_{yz}^{(2)}(x, 0^-) = \tau_0 \qquad |x| \le l \tag{13}$$

$$\left.\begin{aligned} D_y^{(1)}(x, 0^+) = D_y^{(2)}(x, 0^-) \\ \phi^{(1)}(x, 0^+) = \phi^{(2)}(x, 0^-) \end{aligned}\right\} |x| \le \infty \tag{14}$$

$$w^{(1)}(x, 0^+) = w^{(2)}(x, 0^-) = 0 \qquad |x| > l \tag{15}$$

$$w^{(k)}(x, y) = \phi^{(k)}(x, y) = 0$$
$$\text{for } (x^2 + y^2)^{1/2} \to \infty \qquad k = 1, 2 \tag{16}$$

Note that all quantities with superscript $k (k = 1, 2)$ refer to the upper half plane and lower half plane. τ_0 is a magnitude of the stress field.

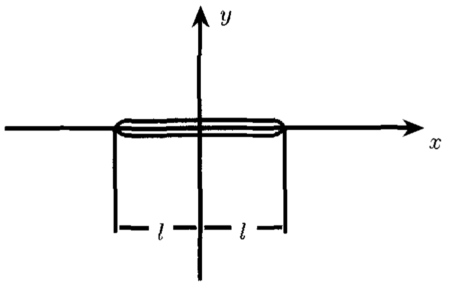

Fig.1 Crack in a piezoelectric material body under anti-plane shear

Substituting Eqs.(9), (10) into Eqs.(1), (2), respectively, using Green-Gauss theorem, it can be obtained[10] that

$$\iint_V \alpha(|x' - x|, |y' - y|)[c_{44}\nabla^2 w(x', y') + e_{15}\nabla^2\phi(x', y')]\mathrm{d}x'\mathrm{d}y' - \int_{-l}^{l} \alpha(|x' - x|, 0)\cdot [\sigma_{yz}(x', 0^+) - \sigma_{yz}(x', 0^-)]\mathrm{d}x' = 0 \tag{17}$$

$$\iint_V \alpha(|x' - x|, |y' - y|)[e_{15}\nabla^2 w(x', y') - \varepsilon_{11}\nabla^2\phi(x', y')]\mathrm{d}x'\mathrm{d}y' - \int_{-l}^{l} \alpha(|x' - x|, 0)\cdot [D_y^c(x', 0^+) - D_y^c(x', 0^-)]\mathrm{d}x' = 0 \tag{18}$$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \le x \le \infty$, $0 \le y \le \infty$ only. Under the applied anti-plane shear load on the unopened surfaces of the crack, the displacement field and the electric displacement satisfy the following symmetry conditions

$$w(x, -y) = -w(x, y) \qquad \phi(x, -y) = -\phi(x, y) \tag{19}$$

Using Eq.(19), we find that

$$\sigma_{yz}(x, 0^+) - \sigma_{yz}(x, 0^-) = 0 \tag{20}$$

$$D_y^c(x, 0^+) - D_y^c(x, 0^-) = 0 \tag{21}$$

Hence the line integrals in Eqs.(17), (18) vanish. By taking the Fourier transform of Eqs.(17), (18) with respect to x', it can be shown that the general solutions of Eqs.(17), (18) are identical to those of the following equations

$$c_{44}\left[\frac{\mathrm{d}^2\bar{w}(s, y)}{\mathrm{d}y^2} - s^2\bar{w}(s, y)\right] + e_{15}\left[\frac{\mathrm{d}^2\bar{\phi}(s, y)}{\mathrm{d}y^2} - s^2\bar{\phi}(s, y)\right] = 0 \tag{22}$$

$$e_{15}\left[\frac{\mathrm{d}^2\bar{w}(s, y)}{\mathrm{d}y^2} - s^2\bar{w}(s, y)\right] - \varepsilon_{11}\left[\frac{\mathrm{d}^2\bar{\phi}(s, y)}{\mathrm{d}y^2} - s^2\bar{\phi}(s, y)\right] = 0 \tag{23}$$

almost everywhere. Here a superposed bar indicates the Fourier transform.

The general solutions of Eqs.(22), (23) satisfying Eq.(16) are, respectively

$$w^{(1)}(x,y)=\frac{2}{\pi}\int_0^\infty A_1(s)\mathrm{e}^{-sy}\cos(xs)\mathrm{d}s$$

$$\phi^{(1)}(x,y)-\frac{e_{15}}{\varepsilon_{11}}w^{(1)}(x,y) \tag{24}$$

$$=\frac{2}{\pi}\int_0^\infty B_1(s)\mathrm{e}^{-sy}\cos(xs)\mathrm{d}s$$

$$w^{(2)}(x,y)=\frac{2}{\pi}\int_0^\infty A_2(s)\mathrm{e}^{sy}\cos(xs)\mathrm{d}s$$

$$\phi^{(2)}(x,y)-\frac{e_{15}}{\varepsilon_{11}}w^{(2)}(x,y) \tag{25}$$

$$=\frac{2}{\pi}\int_0^\infty B_2(s)\mathrm{e}^{sy}\cos(xs)\mathrm{d}s$$

where $A_1(s)$, $B_1(s)$, $A_2(s)$, $B_2(s)$ are to be determined from the boundary conditions.

The stress field and the electric displacement, according to Eqs.(9), (10), are given by, respectively

$$\tau_{yz}^{(1)}(x,y)=\frac{2}{\pi}\int_0^\infty[-\mu sA_1(s)-e_{15}sB_1(s)]\mathrm{d}s\cdot$$

$$\int_0^\infty \mathrm{d}y'\int_{-\infty}^\infty[\alpha(|x'-x|,|y'-y|)+$$

$$\alpha(|x'-x|,|y'+y|)]\mathrm{e}^{-sy'}\cos(sx')\mathrm{d}x' \tag{26}$$

$$D_y^{(1)}(x,y)=\frac{2}{\pi}\int_0^\infty\varepsilon_{11}sB_1(s)\mathrm{d}s\int_0^\infty\mathrm{d}y'\cdot$$

$$\int_{-\infty}^\infty[\alpha(|x'-x|,|y'-y|)+$$

$$\alpha(|x'-x|,|y'+y|)]\mathrm{e}^{-sy'}\cos(sx')\mathrm{d}x' \tag{27}$$

$$\tau_{yz}^{(2)}(x,y)=\frac{2}{\pi}\int_0^\infty[\mu sA_2(s)+e_{15}sB_2(s)]\mathrm{d}s\cdot$$

$$\int_0^\infty \mathrm{d}y'\int_{-\infty}^\infty[\alpha(|x'-x|,|y'-y|)+$$

$$\alpha(|x'-x|,|y'+y|)]\mathrm{e}^{-sy'}\cos(sx')\mathrm{d}x' \tag{28}$$

$$D_y^{(2)}(x,y)=-\frac{2}{\pi}\int_0^\infty\varepsilon_{11}sB_2(s)\mathrm{d}s\cdot$$

$$\int_0^\infty \mathrm{d}y'\int_{-\infty}^\infty[\alpha(|x'-x|,|y'-y|)+$$

$$\alpha(|x'-x|,|y'+y|)]\mathrm{e}^{-sy'}\cos(sx')\mathrm{d}x' \tag{29}$$

where $\mu=c_{44}+e_{15}^2/\varepsilon_{11}$.

Using Eq.(6) for $\alpha(|X'-X|)$, we carry out integrations on x' and y'. To this end we consider the following integrals[18]

$$I_1=\int_{-\infty}^\infty\exp(-px'^2)\begin{Bmatrix}\sin\xi(x'+x)\\ \cos\xi(x'+x)\end{Bmatrix}\mathrm{d}x'$$

$$=(\pi/p)^{1/2}\exp(-\xi^2/4p)\begin{Bmatrix}\sin(\xi x)\\ \cos(\xi x)\end{Bmatrix} \tag{30}$$

$$I_2=\int_0^\infty\exp(-py'^2-\gamma y')\mathrm{d}y'=\frac{1}{2}(\pi/p)^{1/2}\cdot$$

$$\exp(\gamma^2/4p)[1-\Phi(\gamma/2\sqrt{p})] \tag{31}$$

$$\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z\exp(-t^2)\mathrm{d}t \tag{32}$$

Hence

$$\tau_{yz}^{(1)}(x,0^+)=-\frac{2}{\pi}\int_0^\infty s[\mu A_1(s)+e_{15}B_1(s)]\cdot$$

$$\mathrm{erfc}(\varepsilon s)\cos(sx)\mathrm{d}s \tag{33}$$

$$D_y^{(1)}(x,0^+)=\frac{2}{\pi}\int_0^\infty\varepsilon_{11}sB_1(s)\mathrm{erfc}(\varepsilon s)\cos(sx)\mathrm{d}s \tag{34}$$

$$\tau_{yz}^{(2)}(x,0^-)=\frac{2}{\pi}\int_0^\infty s[\mu A_2(s)+e_{15}B_2(s)]\cdot$$

$$\mathrm{erfc}(\varepsilon s)\cos(sx)\mathrm{d}s \tag{35}$$

$$D_y^{(2)}(x,0^-)=-\frac{2}{\pi}\int_0^\infty\varepsilon_{11}sB_2(s)\mathrm{erfc}(\varepsilon s)\cos(sx)\mathrm{d}s \tag{36}$$

where $\varepsilon=\dfrac{a}{2\beta}$, $\mathrm{erfc}(z)=1-\Phi(z)$.

To solve the problem, the jump functions of the displacements and the electric potentials across the crack surface are defined as follows

$$f_w(x)=w^{(1)}(x,0^+)-w^{(2)}(x,0^-) \tag{37}$$

$$f_\phi(x)=\phi^{(1)}(x,0^+)-\phi^{(2)}(x,0^-) \tag{38}$$

Substituting Eqs.(24) and (25) into Eqs.(37), (38), and applying the Fourier transform, it can be obtained that

$$\bar{f}_w(s)=A_1(s)-A_2(s) \tag{39}$$

$$\bar{f}_\phi(s)=\frac{e_{15}}{\varepsilon_{11}}[A_1(s)-A_2(s)]+B_1(s)-B_2(s) \tag{40}$$

Substituting Eqs.(33)~(36) into Eqs.(13), (14), it can be obtained that

$$-\mu[A_1(s)+A_2(s)]-e_{15}[B_1(s)+B_2(s)]=0 \tag{41}$$

$$B_1(s)+B_2(s)=0 \tag{42}$$

$$\frac{e_{15}}{\varepsilon_{11}}[A_1(s)-A_2(s)]+B_1(s)-B_2(s)=0 \qquad (43)$$

By solving four Eqs.(37), (41)~(43) with four unknown functions and applying the boundary conditions (13)~(15), it can be obtained that

$$\frac{1}{\pi}\int_0^\infty s\bar{f}_w(s)\mathrm{erfc}(\varepsilon s)\cos(sx)\mathrm{d}s=-\frac{\tau_0}{c_{44}} \quad |x|\le l \qquad (44)$$

$$\frac{1}{\pi}\int_0^\infty \bar{f}_w(s)\cos(sx)\mathrm{d}s=0 \quad |x|>l \qquad (45)$$

Since the only difference between the classical and the non-local equations is in the introduction of the function $\mathrm{erfc}(\varepsilon s)$, it is logical to utilize the classical solution to convert the system (44), (45) to an integral equation of the second kind that is generally better behaved. For $a=0$, then $\mathrm{erfc}(\varepsilon s)=1$ and Eqs.(44), (45) reduce to the dual integral equations for the same problem in classical piezoelectric materials. To determine the unknown function $\bar{f}_w(s)$, the dual-integral Eqs.(44), (45) must be solved.

4 SOLUTION OF THE DUAL INTEGRAL EQUATIONS

The dual integral equations (44), (45) can not be transformed into the second Fredholm integral equation, because the kernel of the second kind Fredholm integral equation in the paper[10] is divergent, which can be written as follows

$$L(x,u)=(xu)^{1/2}\int_0^\infty tk(\varepsilon' t)\mathrm{J}_0(xt)\mathrm{J}_0(ut)\mathrm{d}t$$

$$0\le x \qquad u\le 1$$

where $\mathrm{J}_n(x)$ is the Bessel function of order n.

$$k(\varepsilon t)=-\Phi(\varepsilon' t)$$

$$\lim_{t\to\infty}k(\varepsilon' t)\ne 0 \quad \text{for } \varepsilon'=\frac{a}{2\beta l}\ne 0$$

l is the length of the crack.

$$\mathrm{J}_0(x)\approx\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{1}{4}\pi\right) \qquad \text{for } x\gg 0$$

The limit of $tk(\varepsilon' t)\mathrm{J}_0(xt)\mathrm{J}_0(ut)$ is not equal to zero for $t\to\infty$. So the kernel $L(x,u)$ in Eringen's paper[10] is divergent. Of course, the dual integral equations (44) and (45) can be considered to be a single integral equation of the first kind with a discontinuous kernel as discussed in Ref.[8]. It is well-known in the literature that integral equations of the first kind are generally ill-posed in the sense of Hadamard, i.e. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. For overcoming the difficult, the Schmidt method[15] is used to solve the dual-integral equations (44), (45). The gap functions of the crack surface displacement are represented by the following series

$$f_w(x)=w^{(1)}(x,0^+)-w^{(2)}(x,0^-)=\sum_{n=1}^\infty a_nP_{2n-2}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1-\frac{x^2}{l^2}\right)^{1/2}$$

$$\text{for} \quad -l\le x\le l \qquad y=0 \qquad (46)$$

$$f_w(x)=w^{(1)}(x,0^+)-w^{(2)}(x,0^-)=0$$

$$\text{for} \quad |x|>l \qquad y=0 \qquad (47)$$

where a_n is unknown coefficients to be determined and $P_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial[18]. The Fourier transformation[19] of Eq.(46) is

$$\bar{f}_w(s)=\sum_{n=1}^\infty a_nG_n\frac{1}{s}\mathrm{J}_{2n-1}(sl) \qquad (48)$$

$$G_n=2\sqrt{\pi}(-1)^{n-1}\frac{\Gamma(2n-1/2)}{(2n-2)!} \qquad (49)$$

where $\Gamma(x)$ and $\mathrm{J}_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eq.(48) into Eqs.(44), (45), respectively, Eq.(45) can be automatically satisfied. Then the remaining Eq.(44) reduces to the form

$$\sum_{n=1}^\infty a_nG_n\int_0^\infty \mathrm{erfc}(\varepsilon s)\mathrm{J}_{2n-1}(sl)\cos(sx)\mathrm{d}s=-\frac{\pi}{c_{44}}\tau_0 \qquad (50)$$

For a large s, the integrands of the Eq.(50) decrease almost exponentially. So they can be evaluated numerically by Filon's method[20]. Equation (50) can now be solved for the coefficients a_n by the Schmidt method[15]. It can be seen in Refs.[12,13].

5 NUMERICAL CALCULATIONS AND DISCUSSION

In fracture mechanics, it is of importance to determine the perturbation stress τ_{yz} and the perturbation electric displacement D_y in the vicinity of the crack's tips. τ_{yz} and D_y along the crack line can be expressed, respectively as

$$\tau_{yz}^{(1)}(x,0)=-\frac{c_{44}}{\pi}\sum_{n=1}^\infty a_nG_n\int_0^\infty \mathrm{erfc}(\varepsilon s)\cdot$$

$$\mathrm{J}_{2n-1}(sl)\cos(xs)\mathrm{d}s \tag{51}$$

$$D_y^{(1)}(x,0) = -\frac{e_{15}}{\pi}\sum_{n=1}^{\infty} a_n G_n \int_0^{\infty} \mathrm{erfc}(\varepsilon s)\cdot \mathrm{J}_{2n-1}(sl)\cos(xs)\mathrm{d}s = \frac{e_{15}}{c_{44}}\tau_{yz}^{(1)}(x,0) \tag{52}$$

So long as $\varepsilon \neq 0$, the semi-infinite integration and the series in Eqs.(51) and (52) are convergent for any variable x. Equations (51) and (52) give finite stresses and electric displacements all along $y = 0$, so there is no stress singularity at the crack tips. However, for $\varepsilon = 0$, we have the classical stress singularity at the crack tips. Within the range $-l < x < l$, $\tau_{yz}^{(1)}/\tau_0$ is very close to unity, and for $x > l$, $\tau_{yz}^{(1)}/\tau_0$ takes finite values diminishing from a finite value at $x = l$ to zero at $x = \infty$. Since $\varepsilon/l > 1/100$ represents a crack length of less than 100 atomic distances[10], and for such submicroscopic sizes other serious questions arise regarding the interatomic arrangements and force laws, therefore, we do not pursue solutions valid at such small crack sizes. The semi-infinite numerical integrals are evaluated easily by Filon's method[20] because of the rapid diminution of the integrands. In all computations, the piezoelectric material is assumed to be the commercially available piezoelectric PZT-4. The material constants of PZT-4 are $c_{44} = 2.56 \times 10^{10}\,\mathrm{N/m^2}$, $e_{15} = 12.7\,\mathrm{c/m^2}$ and $\varepsilon_{11} = 64.6\times10^{-10}\mathrm{c/Vm^2}$, respectively. The results of the stress field and the electric displacement field are plotted in Figs.2 to 9.

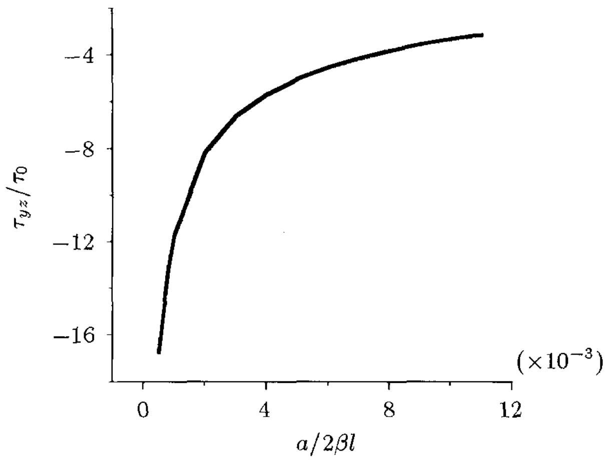

Fig.2 The stress at the crack tip versus $a/2\beta l$

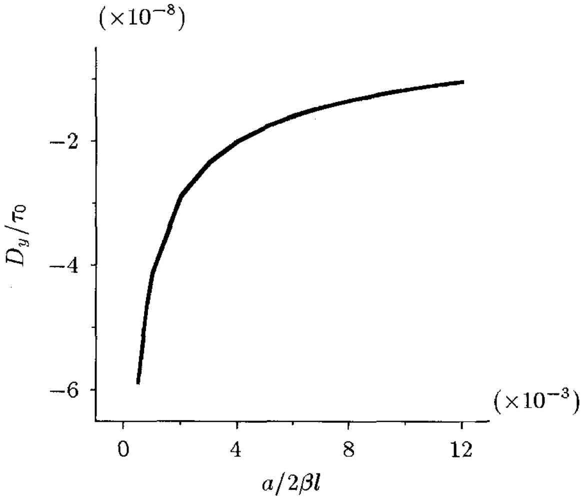

Fig.3 The electric displacement at the crack tip versus $a/2\beta l$

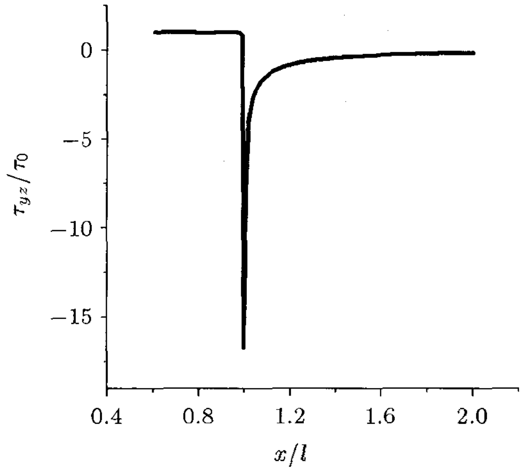

Fig.4 The stress along the crack line versus x/l for $a/2\beta l = 0.000\,5$

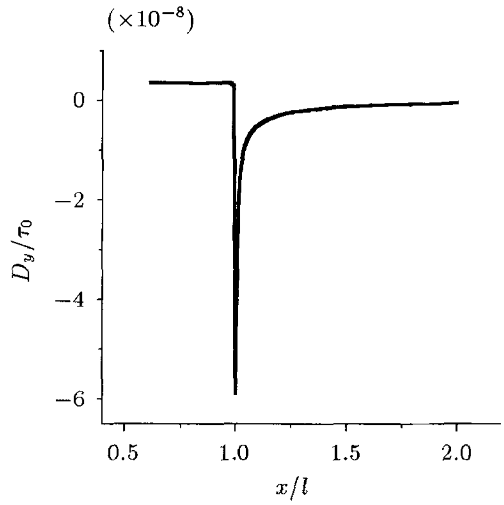

Fig.5 The electric displacement along the crack line versus x/l for $a/2\beta l = 0.000\,5$

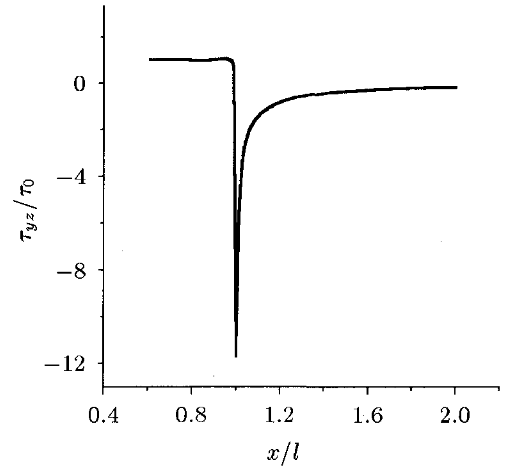

Fig.6 The stress along the crack line versus x/l for $a/2\beta l = 0.001$

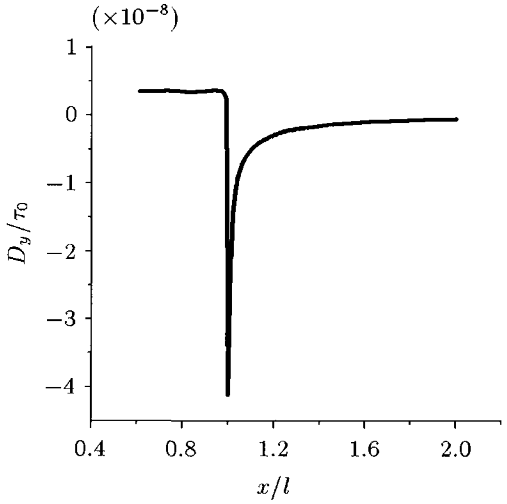

Fig.7 The electric displacement along the crack line versus x/l for $a/2\beta l = 0.001$

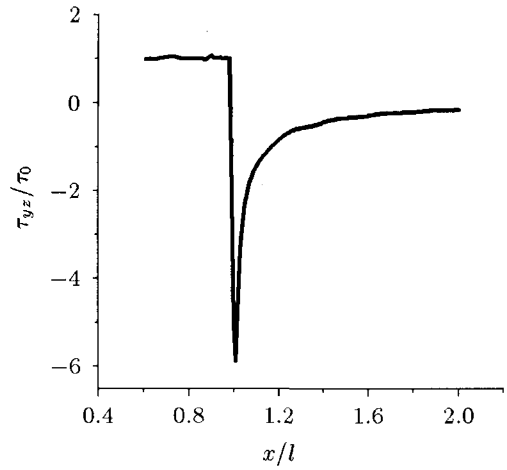

Fig.8 The stress along the crack line versus x/l for $a/2\beta l = 0.005$

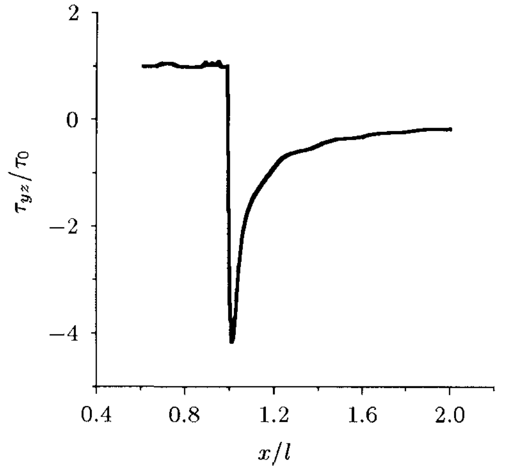

Fig.9 The stress along the crack line versus x/l for $a/2\beta l = 0.01$

The following observations can be made:

(1) For $\varepsilon \neq 0$, it can be proved that the semi-infinite integration in Eqs.(51) and (52) and the series in Eqs.(51), (52) are convergent for any variable x. So the stress and the electric displacement take finite values all along the crack line. Contrary to the classical piezoelectric theory solution, it is found that no stress and electric displacement singularity is present at the crack tip, and also the present results converge to the classical ones in regions far away from the crack tip. The maximal stress does not occur at the crack tip, but slightly away from it. This phenomenon has been thoroughly substantiated in Ref.[22]. The distance between the crack tip and the maximum stress point is very small, and it depends on the crack length and the lattice parameter.

(2) The stress at the crack tip becomes infinite as the atomic distance $a \to 0$. This is the classical continuum limit of square root singularity. This can be shown in Eqs.(44), (45). For $a \to 0$, $\mathrm{erfc}(\varepsilon s) = 1$, Eqs.(44), (45) may reduce to the dual integral equations for the same problem of classical piezoelectric materials. These dual integral equations can be solved by using the singular integral equation for the same problem of the local piezoelectric materials problem. However, the stress and the electric displacement singularities are present at the crack tip in the local piezoelectric materials problem as is well known.

(3) For $a/\beta =$ constant, viz., the atomic distance does not change, the value of the stress concentrations (at the crack tip) increases with the increase of the crack length ($a/2\beta l$ will become smaller with

the increase of the crack length l). Experiments also indicate that the piezoelectric materials with smaller cracks are more resistant to fracture than those with larger cracks.

(4) The significance of this result is that the fracture criteria are unified at both the macroscopic and microscopic scales, viz., it may solve the problem of any scale cracks (that is, any value of $a/2\beta l$).

(5) The stress concentration occurs at the crack tip as stated in Refs.[9,10], and this is given by

$$\tau_{yz}(l,0)/\tau_0 = -c_3/\sqrt{a/2\beta l} \tag{53}$$

where c_3 converges to $c_3 = 0.37$.

(6) The dimensionless stress field is found to be independent of the material parameters. They just depend on the length of the crack and the lattice parameter. However, the electric displacement is found to depend on the loads, the length of the crack and the lattice parameter.

(7) The results of the stress and the electric displacement at the crack tip tend to decrease with the increase of the lattice parameter as shown in Figs.2 and 3.

(8) The electric displacement for the permeable crack conditions is much smaller than the results for the impermeable crack conditions as shown in Figs.3 and 5.

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